THE HARDY-LITTLEWOOD CIRCLE METHOD

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İzmir Institute of Technology, Arithmetic seminar, 9/12/2025

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• We also call these series and their generalizations exponential sums, and when the sum is finite they are called trigonometric polynomials.

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• It allows study of diophantine problems via Fourier series, but also helps with problems of harmonic analysis with an underlying arithmetic structure.

• This method has a century of history behind it now, it has been developed and sharpened by some of the best mathematicians of 20th century. As it stands, it is quite technical, and connected to many other deep issues of mathematics. This talk will be only an introduction, a motivation to this method. We will emphasize how it emerges from observing the behaviour of exponential sums.

• Consider a diophantine equation and its integer solutions, which we can simply consider as the zero set of a polynomial with integer coefficients

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• With $\mathbb{T} = [0, 1]$, we can count this solutions set as follows

$$\int_{\mathbb{T}} \sum_{(n_1,n_2,\ldots,n_s)\in\mathbb{Z}^s} e^{2\pi i P(n_1,n_2,\ldots,n_s)x} dx = \sum_{(n_1,n_2,\ldots,n_s)\in\mathbb{Z}^s} \int_{\mathbb{T}} e^{2\pi i P(n_1,n_2,\ldots,n_s)x} dx$$

$$= \left|\{(n_1, n_2, \ldots, n_s) \in \mathbb{Z}^s : P(n_1, n_2, \ldots, n_s) = 0\}\right|.$$

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• One very famous problems thus studied is the Waring problem, which is very old (1770). It asks how many kth powers are needed to write every positive number as a sum.

• For example every positive number can be written as a sum of at most 4 squares. (The Lagrange theorem)

- We observe that this is counting

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• If this set is nonempty, then N can be written as a sum of s kth powers. Moreover, it gives in how many ways we can write it as a sum of s kth powers. If for every N this set is nonempty, then we see that all positive numbers can be written as a sum of s kth powers.

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• We can relate this problem to exponential sums via the polynomial $P(n_1, n_2, \ldots, n_s) = n_1^k + n_2^k + \ldots + n_s^k - N$.

- Then

$$\left|\{(n_1, n_2, \ldots, n_s) \in \mathbb{Z}_+^s : n_1^k + n_2^k + \ldots + n_s^k = N\}\right|$$

$$= \int_{\mathbb{T}} \sum_{(n_1, n_2, \ldots, n_s) \in \mathbb{Z}_+^s} e^{2\pi i [n_1^k + n_2^k + \ldots + n_s^k - N] x} dx$$

$$= \int_{\mathbb{T}} \Big[\sum_{n=0}^{\lfloor N^{1/k} \rfloor} e^{2\pi i n^k x} \Big]^s e^{-2\pi i N x} dx.$$

- For another closely related problem of counting

$$|\{(n_1, n_2, \ldots, n_{2s}) \in [0, N]^{2s} : n_1^k + \ldots + n_s^k = n_{s+1}^k + \ldots + n_{2s}^k\}|$$

$$= \int_{\mathbb{T}} \sum_{(n_1, n_2, \ldots, n_{2s}) \in [0,N]^{2s}} e^{2\pi i [n_1^k + \ldots + n_s^k - n_{s+1}^k - \ldots - n_{2s}^k] x} dx$$

$$= \int_{\mathbb{T}} \Big| \sum_{n=0}^{N} e^{2\pi i n^k x} \Big|^{2s} dx.$$

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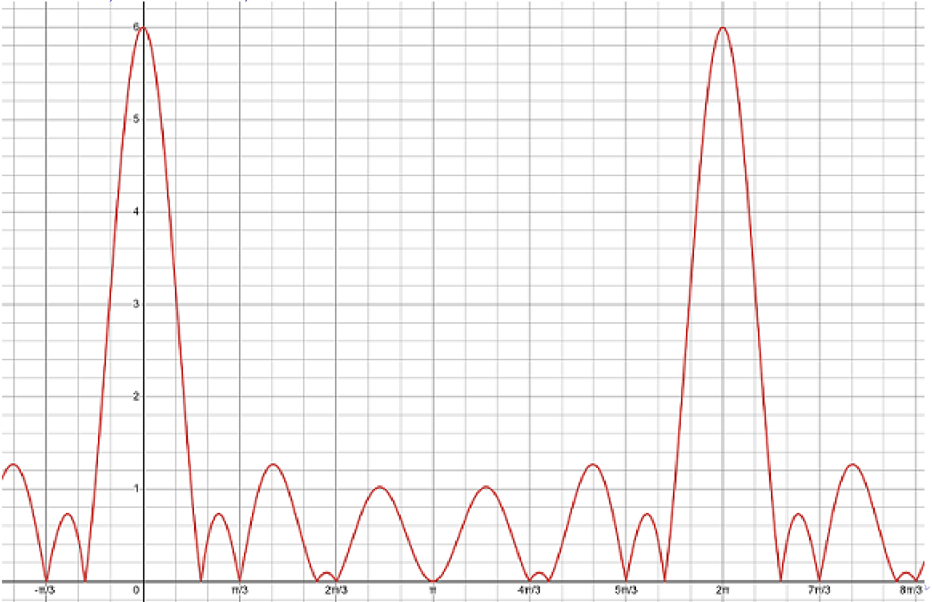
• We will now look at graphs of this sum to understand its behavior. As it is complex-valued, we can graph its real and imaginary parts separately. In any case both behave similarly. So we will graph

$$\Big|\sum_{n=0}^{N} \cos n^k x\Big|,$$

for values $N = 5, 12$ and $k = 1, 2, 3$. Then discuss why this behaviour emerges.

$\cos nx$, $N = 5$, $k = 1$



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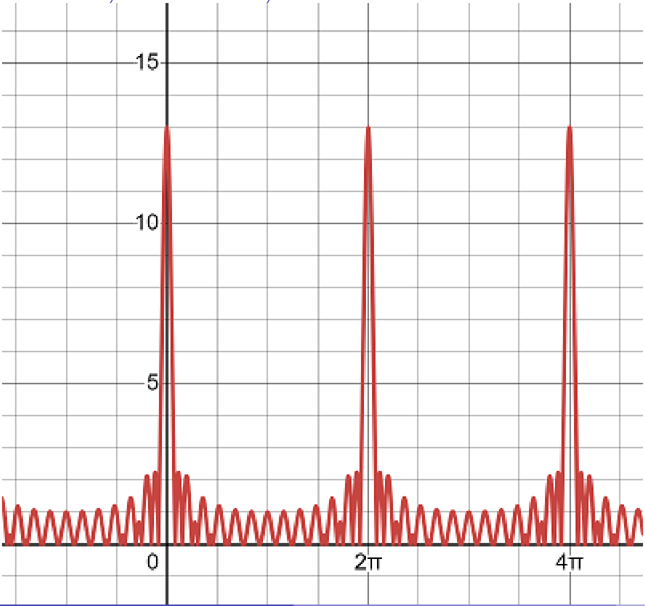
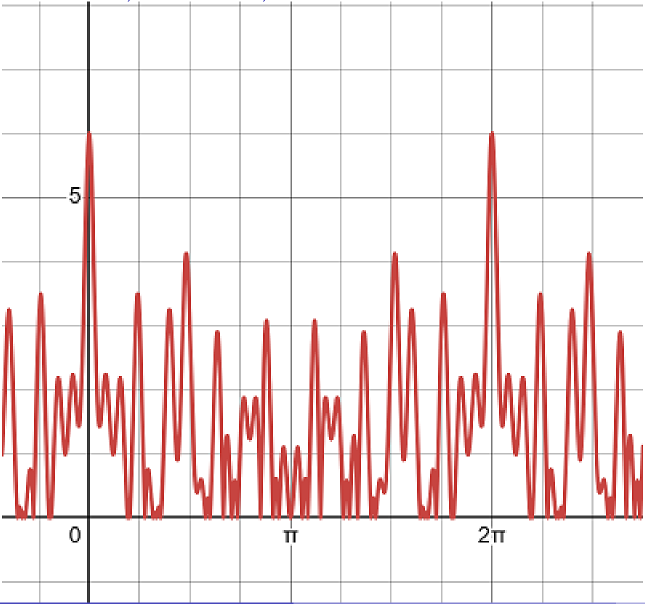


$\cos n^2 x$, $N = 5$, $k = 2$



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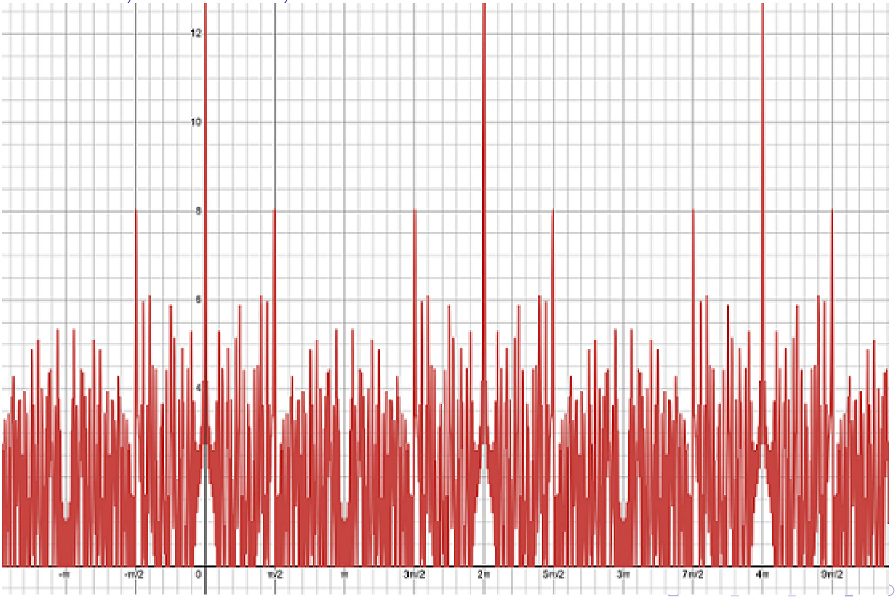
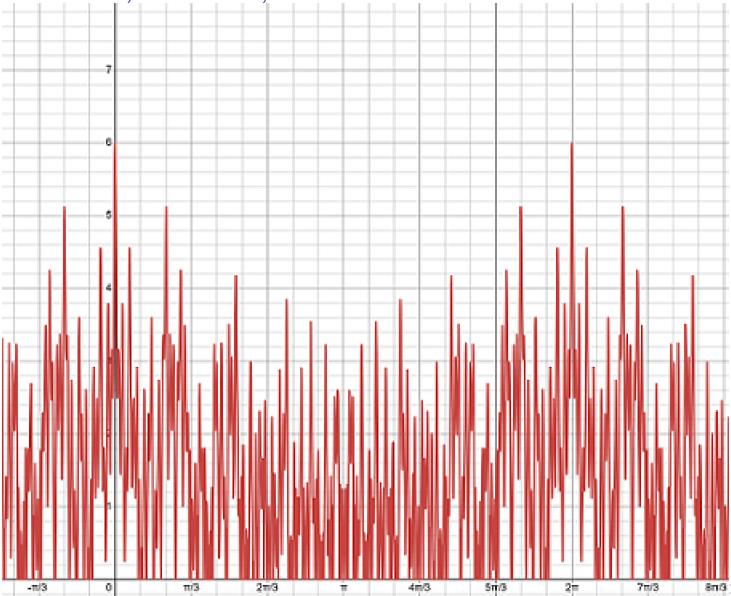


$\cos n^3 x, N = 5, k = 3$



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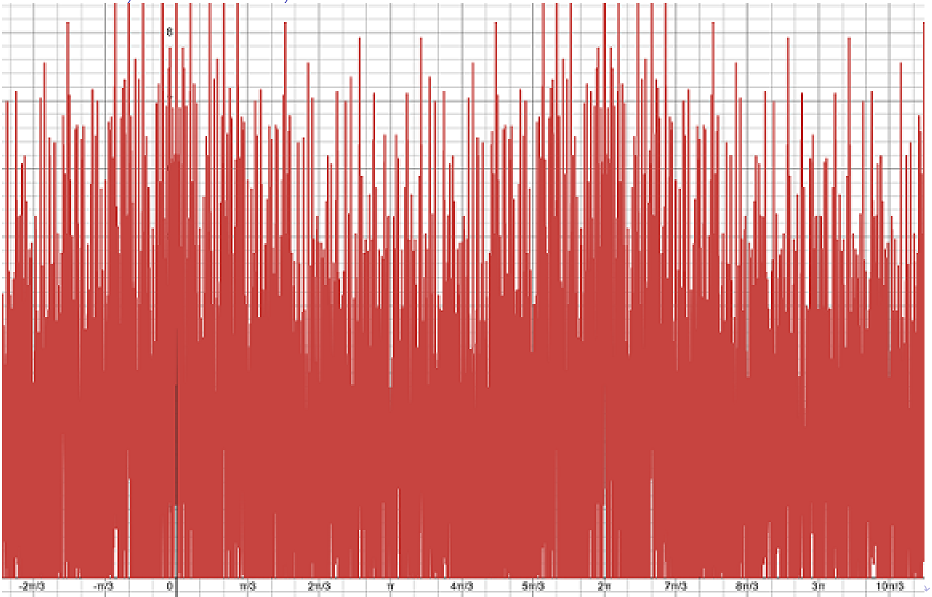
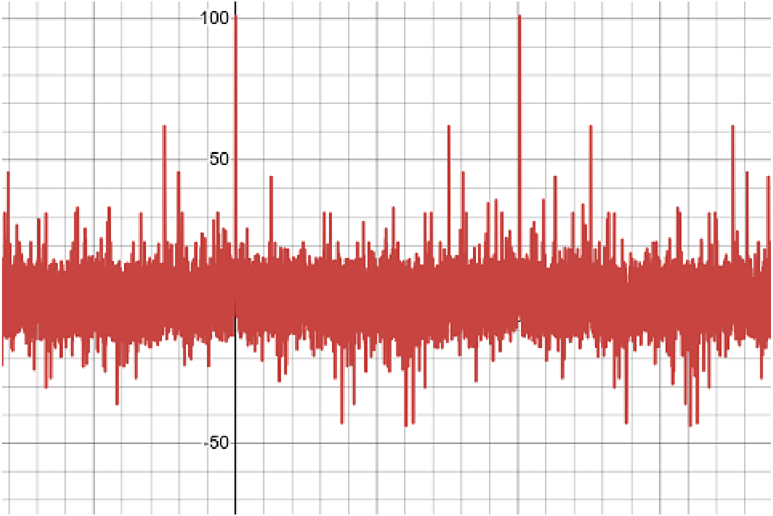
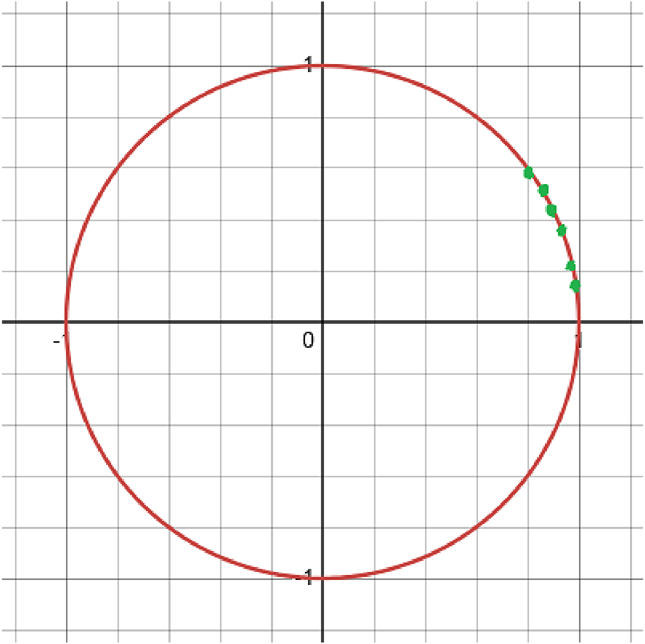


- Lastly we also graph

$$y = \sum_{n=0}^{100} \cos \frac{n^2 x}{16}$$


100
50
-50

• Why this behaviour emerges? Each $e^{i\theta}$ lies on the unit circle. Suppose we take six such points, $e^{\theta_j}, 0 \leq j \leq 5$. If these are equidistributed on the circle, their sum involves a lot of cancellation, and is small compared to N. If they lie all on the same part of the circle, their sum will have little cancellation, and the sum will be comparable to N.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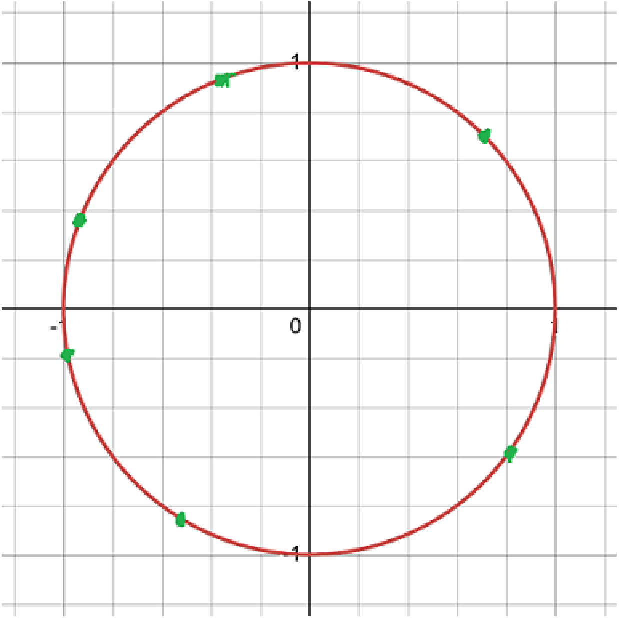
-1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$k = 1$

• For $k = 1$, there is a closed form expression, and it explains the behaviour. Indeed

$$\sum_{n=-N}^{N} e^{inx}$$

is known as the Dirichlet kernel, and it can easily be calculated using the well known formula

$$1 + u + u^2 + u^3 + \ldots + u^n = \frac{u^{n+1} - 1}{u - 1}, \quad u \neq 1.$$

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• We pick $u = e^{ix}$ and $u = e^{-ix}$ and apply this formula. Eventually we get

$$\sum_{n=-N}^{N} e^{inx} = \frac{\sin(N + 1/2)x}{\sin x/2}$$

- We can then use

$$\sum_{n=-N}^{N} e^{inx} = -1 + 2\sum_{n=0}^{N} \cos nx,$$

to get

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• For x small, $\sin x/2 \sim x/2$, but $\sin(N+1/2)x$ can be close to $1, -1$. So we get $1/x$ like behavior. Since this is a periodic function, this behaviour repeats at every integer. We therefore write, for $\|x\|$ denoting distance to the nearest integer

$$\Big|\sum_{n=0}^{N} \cos nx\Big| \lesssim \min\{\|x\|^{-1}, N+1\}.$$

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• But this is a very special situation. For any sum of characters

$$\sum_{n=0}^{N} e^{ia_n x}, \qquad a_n \in \mathbb{Z},$$

by picking $|x| < \left(10 \max_{0\le n\le N}\{a_n\}\right)^{-1}$, we have

$$\Big|\sum_{n=0}^{N} e^{ia_n x}\Big| > \sum_{n=0}^{N} \cos a_n x \ge \sum_{n=0}^{N} \cos 1/10 \ge \frac{99}{100} N.$$

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• This indeed establishes existence of a peak at 0. For $|x| < \left(10 \max_{0\le n\le N}\{a_n\}\right)^{-1}$, the exponetial sum is certainly larger in size than $99N/100$.

Average value

- Also

$$\Big\| \sum_{n=0}^{N} e^{ia_n x} \Big\|^2_{L^2([0,2\pi])} = \int_0^{2\pi} \Big| \sum_{n=0}^{N} e^{ia_n x} \Big|^2 dx$$

$$= \int_0^{2\pi} \sum_{n=0}^{N} e^{ia_n x} \cdot \overline{\sum_{m=0}^{N} e^{ia_m x}} dx = \int_0^{2\pi} \sum_{n=0}^{N} \sum_{m=0}^{N} e^{i(a_n - a_m)x} dx$$

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- So average value of $\Big| \sum_{n=0}^{N} e^{ia_n x} \Big| \approx \sqrt{N}$. These two heuristics are important. We can see the peak at zero in all our graphs. Also we see for $k > 1$ that graphs move up to the value $\sqrt{N}$ quite often before going down.

Irrational points

• For $k > 1$, there is no closed form, and things get very difficult. The two facts uncovered above, that is the peak at zero, and the average value are very helpful, but we need a more complete understanding. To understand why things get out of hand, let us consider various points in $[0, 1]$.

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• Indeed equidistibution is equivalent to having significant cancellation:

$$\sum_{n=0}^{N} e^{ia_n x} = o(N).$$

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• These constitute the minor arcs part in the Hardy-Littlewood circle method.

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• Another very basic fact is: we can regard an exponential sum

$$\sum_{n=0}^{N} a_n e^{2\pi i \theta_n x}$$

as essentially constant on intervals of size $\left[\max_{0 \leq n \leq N}\{\theta_n\}\right]^{-1}$. This is by the uncertainty principle.

- So we can regard the sum

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• So when investigating this sum it suffices to consider rational points p/q with $q \leq N^{k/2}$.

• With this at hand, let us pick $k = 2$, and see what happens.

Gauss sums

• Pick $q \leq N$. Then n^2, for the numbers $n = 0, 1 \ldots q-1$ gives us quadratic residues $\mod q$, and then for larger n this repeats. Thus we have

$$\sum_{n=0}^{N} e^{2\pi i n^2 \frac{p}{q}} \approx \frac{N}{q} \sum_{r=0}^{q-1} e^{2\pi i r^2 \frac{p}{q}}$$

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• This last sum then satisfies the size bound $\sqrt{q}$, by theory of Gauss sums. Thus

$$\left| \sum_{n=0}^{N} e^{2\pi i n^2 \frac{p}{q}} \right| \lesssim N/\sqrt{q}.$$

So unless q is very small, even for $q \geq N^{\epsilon}$, this means significant cancellation, and equidistribution.

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• Let us see what happens for rational points of small denominator.

Rationals of small denominator, $k = 1$

- Let us investigate the point $x = 1/3$. Then we have

$$\sum_{n=0}^{N} e^{2\pi i n/3} = \sum_{r=0}^{2} \sum_{\substack{n \equiv r \\ n \in [0,N]}} e^{2\pi i n/3} = \sum_{r=0}^{2} \sum_{\substack{n \equiv r \\ n \in [0,N]}} e^{2\pi i r/3}$$

$$= \sum_{r=0}^{2} |\{n \in [0,N] : n \equiv r\}| e^{2\pi i r/3}.$$

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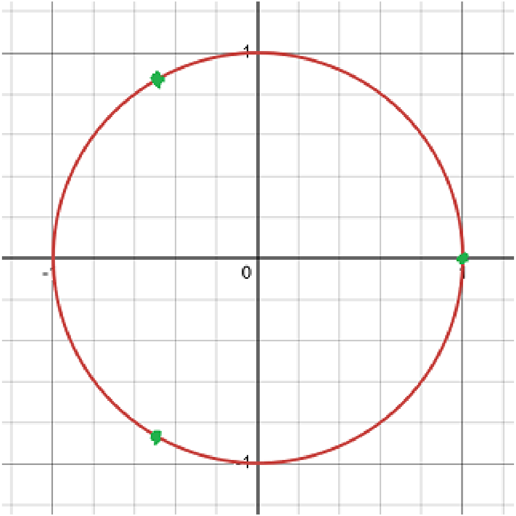
- We have

$$\frac{N-1}{3} \leq |\{n \in [0, N] : n \equiv r\}| \leq \frac{N}{3} + 1.$$

- So we write

$$= \frac{N}{3}\sum_{r=0}^{2} e^{2\pi i r/3} + O(1) = O(1).$$



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• Now let's investigate this for $k = 2$. If $r \equiv 1, 2 \mod 3$, then $r^2 \equiv 1 \mod 3$. For $r \equiv 0$ then $r^2 \equiv 0$. This destroys the structure that gives rise to cancellation. We get

$$\sum_{n=0}^{N} e^{2\pi i n^2/3} = \sum_{r=0}^{2} \sum_{\substack{n \equiv r \\ n \in [0,N]}} e^{2\pi i n^2/3} = \sum_{\substack{n \equiv 0 \\ n \in [0,N]}} 1 + \sum_{r=1,2} \sum_{\substack{n \equiv 0 \\ n \in [0,N]}} e^{2\pi i/3}$$

$$= \frac{N}{3} + \frac{2N}{3} \frac{\sqrt{3}i - 1}{2} + O(1) = \frac{N}{\sqrt{3}} i + O(1).$$

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$$= \frac{N}{3} + \frac{2N}{3} \frac{\sqrt{3}i - 1}{2} + O(1) = \frac{N}{\sqrt{3}} i + O(1).$$

• So we do not have much of a cancellation, that is we do not have $o(N)$ behavior. Similar behavior happens for larger k as well.

The Roadmap

• This investigation yields a blueprint, a roadmap to setting up the method of investigation that we now call the Hardy-Littlewood circle method.

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• Since main contribution comes from the rationals of small denominator, we isolate neighborhoods of these, and call these the major arcs, as they are the major contributor, although their individual and total length is small.

• Irrationals and rationals of large denominator contribute less to the integral, and they are now lumped into the remaining arcs on the torus, called minor arcs, because their contribution is minor, although their length is greater.

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• Now lets see this roadmap in mathematical language.

Defining major and minor arcs.

- Let $v = 1/100$. Then for $1 \leq a \leq q \leq N^v$ we define the arcs

$$\mathfrak{M}(a, q) := \{\alpha : |\alpha - \frac{a}{q}| \leq N^{v-k}\}.$$

These are the major arcs, and $\mathfrak{M}$ stands for their union. There are about N^{2v} of these arcs, each with length $2N^{v-k}$. So total length is $\approx N^{3v-k}$, which is an incredibly tiny part of the torus.

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• The remaining pieces of the torus constitute the minor arcs. We have

$$\left|\frac{a}{q} - \frac{a'}{q'}\right| \geq \frac{1}{qq'} \geq N^{-2v}.$$

So length of the minor arc between two major arcs $\mathfrak{M}(a, q), \mathfrak{M}(a', q')$, is about N^{-2v} and much longer than these major arcs.

Minor arcs

• On minor arcs we have quantified the distance to rationals of small denominator, so we can demonstrate equidistribution and decay efficiently, that is quantitatively. The fundamental idea here is Weyl differencing. It reduces $k > 1$ situtations iteratively to $k = 1$. Lets see this fundamental idea.

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$$\Big|\sum_{n=0}^{N} e^{in^2x}\Big|^2 = \sum_{m=0}^{N}\sum_{n=0}^{N} e^{ix(n^2-m^2)}$$

Let $n = m + h$. Then

$$= \sum_{h=-N}^{N}\sum_{m=-h\vee 0}^{N-h\wedge N} e^{ix(2mh+h^2)} \leq \sum_{h=-N}^{N}\Big|\sum_{m=0}^{M} e^{ix2mh}\Big| \leq \sum_{h=-N}^{N} \min\{\|2hx\|, N+1$$

• This argument can be iterated for higher powers k.

Major arcs

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• For major arcs, we know that at the center points of these there can be less cancellation. But the advantages are that there are only N^{2v} of these points, and the arcs are very short.

• The contribution of each major arc $\mathfrak{M}(a,q)$ can be written as a perturbation of contribution of the center point a/q. This gives a product of contribution from a/q and an integral. Then summing over a/q gives what we call the singular sum, and the singular integral. Lets see these in mathematical language.

- We define

$$S(q,a) := \sum_{m=1}^{q} e^{2\pi i a m^k/q} \qquad v(\beta) := \sum_{m=1}^{N^k} \frac{1}{k} m^{k-1} e^{2\pi i \beta m}.$$

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- Then pointwise for $\alpha \in \mathfrak{M}(q,a)$ we have

$$f(\alpha) := \sum_{m=1}^{N} e^{2\pi i m^k \alpha} = q^{-1} S(q,a) v(\alpha - a/q) + O(N^{2v}).$$

• When we integrate f over major arc $\mathfrak{M}(q,a)$, and then sum over these arcs we obtain

$$\int_{\mathfrak{M}} f(\alpha)^s e^{-2\pi i\alpha n} d\alpha = \mathfrak{S}(n)J(n) + Error$$

where we have the singular series

$$\mathfrak{S}(n) := \sum_q \sum_a (q^{-1}S(q,a))^s e^{-2\pi i a n/q}$$

and the singular integral

$$J(n) = \int_{-N^{v-k}}^{N^{v-k}} v(\beta)^s e^{-2\pi i\beta n} d\beta.$$

- Then aim is to prove that $\mathfrak{G}(n) \gtrsim 1$, and the asymptotics

$$J(n) = cN^{s-k} + Error$$

for s as small as possible. It is conjectured that $s \geq k + 1$ is sufficient. For large k by works of Bourgain $s \sim k^2$ is known. These recent advances came from the Fourier decoupling theory.

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R.C.Vaughan, The Hardy-Littlewood method, second edition, Cambridge University Press, 1997

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• THANK YOU